A COMPARISON OF PATTERN-BASED AND EQUATION-SOLVING APPROACHES TO ALGEBRA

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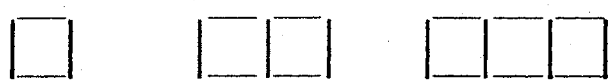
The National Statement on Mathematics for Australian Schools advises that algebra learning begins with the study of sequences and patterns leading to their description as algebraic rules relating dependent and independent variables. This study assessed the success of 512 students in seven schools at year levels 7 to 10 in recognizing and describing algebraic rules relating two variables. When given a relationship described by a table of values for two variables, the majority of these students were unable to write an algebraic rule of the form $y = ax + b$. Comparison of the test results from schools using different approaches to algebra suggests that the pattern-based approach (as implemented at the schools taking part in the study) was no more helpful than traditional approaches.

In *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990), it is suggested that algebraic thinking should be based on initial experiences with patterns and sequences. It is expected that in the primary years children will investigate patterns and sequences and make generalizations about them in everyday language. For example, a pattern of matches:



could be described by children as "You start with four for the first square and then add on three for each extra square" (*National Statement*, p. 191). In the secondary school students learn how to express such generalizations mathematically using algebraic symbolism. Some of the first algebraic skills to be developed, according to the *Statement*, are (a) describing in words relationships and rules for generating elements in a pattern or sequence, (b) expressing one of the rules algebraically, and (c) generating further elements of a pattern from a given verbal or algebraic rule.

Introducing algebra in this way as a language for expressing relationships between two variables, which we will call the "pattern-based approach", represents a clear break with tradition. It seems likely to facilitate the later study of formulas and functions. It is aesthetically pleasing to teachers, and students can make good use of concrete materials for building designs like the two-dimensional pattern above. As well as being endorsed in the *National Statement*, the pattern-based approach appears in the *RIME Lesson Pack* (Lowe and Lovitt, 1984), in the New South Wales mathematics syllabus (NSW Board of Secondary Education, 1989) and in some currently-used secondary school textbooks (e.g., Blane & Booth, 1991; Tomlinson, Ardley, Mottershead, Thompson and Wrightson, 1987).

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Other textbooks in common use (e.g., McLeod, Ganderton, Creeley and Tanti, 1988; Schnabl, Schnabl and Wagstaff, 1985) continue to introduce algebra as a technique for finding an unknown number, in which the earliest activities for students are manipulating expressions and solving equations.

The two approaches are based on two different meanings of algebraic letters. In the pattern-based approach, letters represent variables with a range of values, and in the more traditional approach, letters stand for specific but unknown numbers. In the United States, the NAEP study (Herscovics, 1989) showed that although most students with one or two years of algebra experience could recognize a simple pattern linking two variables (e.g. "add 7") from a table of values, about one-quarter of them were unable to write the corresponding equation (e.g. $y = x + 7$). The systematic research of the British CSMS study (Kuchemann, 1981) described four levels of understanding of algebra based on the interpretation of letters demanded by an item. For items at the lowest levels, where most students were successful, algebraic letters could be ignored, immediately evaluated or interpreted as objects. Items at level 3 required the use of letters as unknown numbers in very simple expressions, but the majority of 13, 14 and 15 year olds could not deal with these items. Items at level 4, in which students were required to interpret a letter as a variable, were dealt with successfully by less than 10% of 15 year olds. Textbooks used by British students were criticized at that time as tending to present algebra "as an afterthought to other work, for example on number patterns where from the child's viewpoint it serves no useful purpose" (Kuchemann, 1981, p. 117).

In view of the results from Britain and USA, it may seem foolhardy of Australian curriculum planners to advocate the introduction of algebra through an approach based on the concept of a variable. However the attractiveness of the approach has been outlined by Pegg and Redden (1990) who point to the opportunity it provides for students to see algebraic notation arising as a natural and useful consequence of expressing generality, the focus on letters as standing for numbers rather than names, and the opportunity it offers for students to see different correct expressions of generalization for the same pattern which can lead to opportunities to establish informally the rules of algebraic manipulation. To illustrate using the example above: students who see the pattern as "four matches for the first square and then three for each extra square" should write the algebraic rule $N = 4 + 3 \times (s - 1)$ where N is the number of matches required for s squares. On the other hand, a student who saw the pattern as "start with one match and add on three for each square" would want to write $N = 1 + 3s$. Comparison of these two correct expressions establishes some aspects of the distributive property.

As frequently happens in curriculum innovation, the recommended change in the method of introducing algebra has not been grounded in any empirical research. No study comparing the effectiveness of the two approaches has been published. The evidence which is presented in this paper suggests that the pattern-based approach may be no better than other approaches for the majority of students.

This paper presents a measure of students' success in recognizing a linear relationship between two variables when given a table of corresponding values of the variables, and in writing it algebraically. In particular, we wanted to find out what percentage of students at various year levels can:

1. interpret a table to read off corresponding values of two variables x and y;
2. fill in a missing entry in the table;

3. use a rule, rather than counting up, for calculating y;
4. write a rule relating the two variables algebraically.

Some students involved in the study had been taught with a pattern-based approach, whereas others had been taught with a traditional approach based on manipulating expressions and solving equations.

PROCEDURE

Two test items were prepared, and included in a classroom test. The items, called A and B in this paper, are shown in Figure 1. The items could be expected to favour the schools where the pattern-based approach was being used. The test was given in seven schools to 512 students in 26 classes from Year 7 to Year 10. Information about textbooks and other materials that had been used for these classes was obtained from the teachers concerned. Students had sufficient time to complete all the questions reported in this paper. All test papers were marked by the researchers. Results and comments were sent to the teachers.

A. Look at the numbers in this table and answer the questions.

x	y
1	5
2	6
3	7
4	8
5	9
6	..
7	..
8	..
..	..

(i) When x is 2, what is y?
(ii) When x is 6, what is y?
(iii) When x is 20, what is y?
(iv) Use algebra symbols to write the rule connecting x and y.

B. Look for the pattern in this table.

First number	0	1	2	3	4	5
Second number	-1	9	19	29	39	..

(i) Work out the missing number.
(ii) Explain how you worked it out.
(iii) Use algebra symbols to write a rule for working out the second number. In your rule, use F to represent the first number and S to represent the second number.

Figure 1: Test Items

RESULTS AND DISCUSSION

Table 1 shows the percentages correct for each part of Item A, and for the missing value and the rule in Item B. Almost all students could read and continue the number patterns in Items A and B, and therefore were correct for A(i) and (ii) and B (i). Working out the value of y when x is 20 (item A(iii)) was more difficult. As may be seen in the table, for both the items, and especially for Item B, many students were unable to write a correct algebraic rule.

Table 1: *Percentage of correct responses to items A and B*

		Item A				Item B	
Year	n	(i) x=2, y=6	(ii) x=6, y=10	(iii) x=20, y=24	(iv)Rule y=x+4	F=5, S=49	Rule y=10x-1
7	179	95%	92%	69%	33%	98%	7%
8	211	99%	97%	89%	74%	98%	47%
9	66	94%	91%	74%	44%	100%	14%
10	56	100%	98%	82%	46%	96%	11%

Results for item A (iii) and (iv)

As shown in Table 1, calculating y when $x = 20$ caused many errors, especially at Year 7 level where 31% were not successful. The most frequent wrong answers over all year levels were 16 (13 students, probably subtracting rather than adding 4) and 23 or 25 (21 students, probably not using any rule but losing track when counting on). Ten students had probably used a direct proportion rule which Stacey (1989) found that many students invent when counting becomes too hard. For example, for $x = 20$, seven students wrote $y = 28$, twice the value of y when $x = 10$. On the basis of Stacey's findings, we believe that more students would have used direct proportion if the value of x in question (iii) had been larger, so that counting was impractical, and if it had been a simple multiple of the value of x in question (ii).

For A (iv), in which students were required to write the rule $y = x + 4$ or equivalent, results overall were disappointing. As shown in Table 1, only one-third of the Year 7 sample wrote a correct rule, and the Year 9 and 10 students were not much better at around 45% correct. Common incorrect attempts to write the rule were xy, $x + y$ and the reversed equation $x = y + 4$. Eleven students (2% of the sample) wrote $x + 4$. At each year level there were two or three students who wrote rules involving 5, (e.g. $y = x + 5$ and $x = 5y$) which we assume to be derived from the first row of the table. Errors in the year 10 sample, for example, include the following:

$$y = 5 - x, \quad xy, \quad x = 1 + 4y, \quad 1x = 5y, \quad x = y, \quad x = y + 4.$$

At all year levels, the forms of many responses reveal lack of understanding of the meaning and use of algebraic notation.

Results for item B

Completing the missing entry in the table for Item B was a simple task for almost all students. However, as may be seen in Table 1, most students in the sample, at all levels, were unable to write an algebraic rule relating the two variables.

As expected, most descriptions of how students had worked out the missing number were attempts to say "I added ten". Many of the incorrect rules are obviously intuitive attempts to write a next-value rule such as "Add ten to the previous number" or "If you increase the first number by 1, then you increase the second number by 10". They include $F = S + 10$, $S = S + 10$ (this would be correct in BASIC programming) and $F + 10 = S$ (written by 64 students), as well as variety of expressions containing either $F + 1$ or $F = 1$ together with $S + 10$ or $S = 10$ (written by 21 students). The incorrect rule $F = S + 10$ was the most common error for the Year 10 classes, accounting for almost half of all attempts to write a rule. It may represent students' naive attempts to translate "Add 10 to the second number" into algebraic symbols, or it may be related to the "reversal error" (see MacGregor, 1991, for a summary of the related literature) which leads students to associate the numeral with the larger of two variables when attempting to write an equation. A few students had focussed on the "9 times" relation which they saw in the second column of the table, and tried to incorporate it into their rules as, for example, $S = F$ ¥ $9 + 4$ (which produces the correct missing value 49).

The results provide documentation of an important difficulty obstructing students' construction of formulas from tables. This is the tendency for students to focus on differences between successive values of the dependent variable, rather than on the relationship connecting dependent and independent variables. For example, many primary and junior secondary students will predict the next number in the sequence 1,4,9,16,25, ... by noticing that the differences between the terms are increasing odd numbers, not by recognising that the numbers are squares. Arzarello, (1991), working with 16-year-olds, described this phenomenon by saying that students' thinking was locked into arithmetic concepts which caused them to search for ways to predict the next number in a table from the value of its predecessor. A similar observation has been made by Clement, Narode and Rosnick (1981) and MacGregor (1991) who also report students' attempts to write such a "next-value rule" algebraically. Pegg et al. (1990) described this as a "completing-the pattern" response and commented that it is a typical early response.

Performance in items A and B related to approach to algebra

Performance varied considerably between schools and classes. For example, on Item A(iv), at year 9 level 73% were successful in one school, whereas in another only 18% were successful. On Item B, in several classes at all levels not more than three students were correct. On the other hand, there were four Year 8 classes, all at the same school, with more than 60% correct. This variation in performance is illustrated in Figure 2 which shows the percentage of students from each class who wrote correct algebraic rules, averaged over Items A and B. Figure 2 also indicates the classes which had been taught using a pattern-based approach. The performance of these classes suggests that they were no more successful than the classes who had learned algebra in other ways. It would be unwise to draw firm conclusions about the effectiveness of the pattern-based approach from this data because of the number of schools (seven) in the sample, the method of selection (teachers' interest in using the test for their classes) and uncontrolled factors. As well as possible differences in teaching style, amount of time spent on algebra, recent revision or practice before the test, and the learning environment, there is the distinct possibility that the choice of text made by a school is related to the general level of academic skills of the students. These inadequacies in the selection of schools for testing and in the control of interfering factors must be taken into account when drawing conclusions from the data. However there is no evidence in the results to indicate that learning algebra through a pattern-based approach equips students better to identify relationships between variables and express them algebraically than does a traditional approach.

Figure 2: Scattergram showing facility averaged over two items for classes taught with different approaches to algebra.

Very few of the verbal explanations for Item B, at all year levels, were written in clear and well-formed English. It may be concluded that describing a mathematical procedure or an algebraic rule verbally was a very difficult task for the majority of these students. Some typical examples of their English descriptions and algebraic rules are shown in Figure 3.

Category	Explanation	Rule
Pattern involving 9	It goes up by 9's. [Yr 7]	S+F
	A pattern of 9 being the last number. [Yr 8]	F+S+10
	Every number that has 9 in it. [Yr 9]	F+10S
	Every 10th number has 9 added on. [Yr 10]	S=10+9F
Add ten	The second the number is counting by ten. [Yr 7] -	
	Bottom line is going up by tens. [Yr 8]	Fx10-1=S
	Add 10. [Yr 9]	F+1=S+10
	10 is between every set of numbers. [Yr 10]	F=S+10
Describe two sequences	I added 10 to each number by the increasing 1 number. [Yr 7	F+10=S
	First number progressed by 1, second number progressed by 10. [Yr 8]	F=-1+F10
	When the first number increases by 1, the second number increases by 10. [Yr 9]	no attempt
	Each first number is added on ten in the second. [Yr 10]	no attempt
Relate two variables	Subtract 1 from the first number multiplied by ten. [Yr 7]	Fx10-1=S
	10 times the top column and take 1. [Yr 8]	S=10F-1
	Multiply the first by 10, subtract 1. [Yr.9]	Fx10-1=S
	Subtracting one from the first number where it is multiplied by 10. [Yr 10]	no attempt

Figure 3: Students' explanations and rules for finding the missing number in item B

CONCLUSIONS

The results highlight the extreme difficulty experienced by many students in trying to describe a numerical relationship in words as well as their difficulty in using algebraic symbols to write a rule. For example, a Year 7 student wrote for Item B, "You cent to you get to 9 and then you cent how much figers are up", and a Year 10 student wrote, "We are follow the first and second number with the distance by 10". These two students, and many others in the sample, appear not to have yet developed the level of English language proficiency necessary for talking and writing about mathematical relations. Even the very simple statement required in Item A - for example, "I would add 4" or "It is 4 more" or "You plus it by 4" - was not produced successfully by half the sample of students. It is hard to believe that the English language competence of students in Australian secondary schools at the present time is insufficient for a majority of students to produce a simple statement such as these. Clearly other factors are operating, but in any case it seems likely that language competence is affecting algebra learning.

The difficulty of the items raises questions about the degree of success that can be hoped for when algebra is introduced via the verbal description of patterns. It was apparent that many students' descriptions of perceived patterns, relationships or rules, whether clearly expressed or not, could not possibly be transformed to an algebraic form without extensive paraphrasing. For example, a Year 8 child wrote, "I saw a progression of 10 per unit of *F*", and another wrote, "1=10, so you just minus one". Both these students wrote the algebraic rule correctly, but clearly had not used their written statements as a basis for translation. It is still an open question whether being required to describe a complex relation in English helps in the construction of its algebraic form (MacGregor, 1990).

It is worth noting that the approach to beginning algebra that had been used for the most successful students was based on the view of algebraic letters as specific unknown numbers rather than variables. In the material which the teachers reported having used in years 7 and 8 (based on units developed by Quinlan, Low, Sawyer and White, 1989), algebraic expressions and equations represent arrangements of concrete objects, some of which are hidden and therefore "unknown". Very little emphasis is given to presenting algebra as a way of describing relationships between two sets of variables. The least successful students (i.e., four classes - two at year 7, one at year 9 and one at year 10 level - in which no student was able to write a correct equation for item B) had used a pattern-based approach.

The success achieved by some classes suggests that students in lower secondary school are capable of learning how to formulate algebraic rules from number patterns, while the great discrepancy between results for different classes and schools suggests that different approaches, teaching materials, teaching styles or the learning environment have a powerful effect.

Two interpretations of the pattern-based approach

A close analysis of the way in which texts such as *Mathematics Today* (Tomlinson et al, 1987) and *Moving Through Maths* (Blane et al, 1991) implement the pattern-based approach, as well as the suggestions in the *National Statement*, reveal different routes from observing a pattern to deriving an equation. These differences are possibly of crucial importance for learners. One common textbook approach takes a geometric design, immediately derives from it a table of values and then seeks an algebraic formula which will produce the numbers in the table. Students may work out the formula in a rote fashion by using the constant difference as the coefficient of x (the independent variable) and then adjusting the values by adding or subtracting a constant. Teachers may not ask students to express their generalizations in words, which might help the relationship make sense. In contrast, the version of the pattern-based approach advocated by the *National Statement* and trialled by Pegg et al. (1990) does not derive the algebraic rule from the table of values. Instead, the geometric features of a pattern or design are directly translated into a statement about the relationship, first in an English sentence and then in algebra. If we use the pattern given at the beginning of this paper as an example, the first version of the pattern based approach immediately constructs the number pattern 1,4,7,10,. . . and a formula starting with $y = 3 \times x + ?$ is sought to generate this pattern of numbers. In the other version, a student may notice that "there are the same number of matches along the top as along the bottom but the number of vertical matches is one more than this." With guidance from the teacher, this observation is linked with the expression $x + x + (x + 1)$, and finally with the formula $y = 3x + 1$. Problems in the transition from a verbal expression to an algebraic rule that have not been resolved are (a) how to help students with poor English skills construct a coherent verbal description, and (b) how to proceed when the verbal description cannot be translated directly to algebra.

There appear to be no other published studies of the effectiveness of introducing algebra via a study of patterns and rules relating two variables and their verbal description. The results reported in this paper suggest that it is a difficult way to approach algebra for the majority of students and that it offers no guarantee of success.

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